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Direct Lyapunov method in the theory of replicator system with entropy-like applications

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"We always have to lay special emphasis on mathematical analogies because the concentration on them can promote the development of science"
A.N.Kolmogorov

Abstract. In this report I summarize the results in the application of Direct Lyapunov method to the classical and generalized replicator systems. Based on this results it is shown that there are exist two types of Lyapunov functions: fitness-like and entropy-like. As example it'll be establish that practically all known entropy measures may be obtain from entropy-like Lyapunov function for replicator systems.

1. LYAPUNOV FUNCTION FOR CLASSICAL REPLICATOR SYSTEMS

First replicator-like equation was introduced by Fisher [4], who immediately recognized certain formal analogies between the mechanistic models introduced by Boltzmann [1] to analyse physical systems, and the selection models proposed by Darwin [3] to explain adaptation in biological systems. By considering the dynamical system which describes the changes in gene frequency of the population which occurs under natural selection, Fisher proved a directionality theorem, which he called the fundamental theorem of natural selection.

The absolute fitness of the zygote $\boldsymbol{A}_i\boldsymbol{A}_j$, denoted as w_{ij} is defined as the relative or fractional number of offspring per unit time that the genotype will produce that will grow to maturity and themselves reproduce.

The state of the genetical system at time t is simply the vector $p(t)=(p_1(t),\ldots,p_n(t))$, which is clearly constrained to lie in the standard simplex σ in n-dimensional Euclidean space $\mathbb{R}^n$:

$\sigma=\{p\in\mathbb{R}^n : p_i\geq 0,\quad i=1,...,n,\ \mathbf{e}^T p=1\}.$

Here and in the sequel, the letter **e** is reserved for a vector of appropriate length, consisting of unit entries

$G=DW$, where $D=diag(d_i)$ is diagonal matrix with $d_i>0$ and W is symmetrical matrix. In this case $\left(\text{hence } \mathbf{e}^T p=\sum_{i=1}^n p_i\right)$. The original Fisher's selection equation of haploid genotypes evolution is, then, written as:

$$\dot{p}_i=p_i\left(\sum_{j=1}^n w_{ij}p_j-\sum_k^n\sum_j^n w_{kj}p_k p_j\right)\quad i=1,...,n \qquad (1)$$

where the dot is derivative w.r.t. time. The population state is then given as a point in simplex σ. Fisher took into account only zygote fitness. In this case $w_{ij}=w_{ji}$ i.e. matrix $W=(w_{ij})=W^T$. The difference between symmetric and nonsymmetrical matrices W is crucial. Indeed in the symmetric case the quadratic form $p(t)^T Wp(t)$ is increasing along trajectories of the selection dynamics (1) – this is the Fundamental Theorem of Selection going back to R. A. Fisher.

Theorem 1 (Fisher [4]) *If $W=W^T$ then the function $p(t)^T Wp(t)$ is strictly increasing with increasing t along any non-stationary trajectory $p(t)$ under (1). Furthermore, any such trajectory converges to a stationary point.* □

Fisher claimed that the theorem was an analog of Boltzmann's principle of entropy increase, and thus to be an analog of the Second Law. These claims have later been re-evaluated.

This theorem is tantamount to saying that mean fitness is an Energy Lyapunov function of the dynamical system (1). Let me recall that a single–valued function which is continuous and has continuous partial derivatives is called the Energy Lyapunov function (ELF) for dynamical system if it is monotonically increasing along the trajectories of this dynamical systems [8]. So, if we know ELF for dynamic system under consideration then we know practically everything about evolution of this system.

In 1969 V.A. Ratner [16] showed that if we take into account also gamete selection when the fitness matrix

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Fisher's theorem is a false. But, it was shown by Pykh [10, 12] that there exists an analog of Fisher's theorem. Let us denote by $\langle\cdot,\cdot\rangle$ the inner product of the two vectors.

Theorem 2. (Pykh [10]) *If matrix G has the next form:* $G = D_1 B D_2$, *where B is symmetrical,* D_1 *and* D_2 *are diagonal matrices and matrix* $D = D_1^{-1} D_2 > 0$ *then the function* $E:\sigma \to \mathbb{R}^1$

$$E(p) = \frac{p^T D_2 B D_2 p}{\langle d, p\rangle^2} \tag{2}$$

is ELF for system (1). □

It is natural to name this function as fitness-like Lyapunov function. In order to formulate the next theorem I need the following definition.

Definition. *Matrix G is called D-dissipative on* $\sigma^0 = \left(x : \sum_{i=1}^{n} x_i = 0, -\hat{p}_i \le x_i \le 1 - \hat{p}_{i,} \quad i = 1,...,n \right)$ *if there exists such diagonal matrix D that* $x^T A D x < 0$ for $x \in \sigma^0$, *and positive D-dissipative if* $D = diag(d_i) > 0$. □

Theorem 3. (Pykh [11]) *If system (1) has the equilibrium point* $\hat{p} \in Int\sigma$ *and the matrix G is positive D-dissipative on* σ^0 *with the matrix* $D^{-1} = diag(d_i^{-1})$ *then function*

$$E_d(p) = \sum_{i=1}^{n} d_i \hat{p}_i \ln \frac{p_i}{\hat{p}_i} - \langle d, \hat{p}\rangle \ln\langle d, p\rangle \tag{3}$$

is ELF for system (1) and equilibrium point $\hat{p}$ is stable in $Int\sigma$.□

It is evident that if $d_i = 1, \quad i = 1,...,n$ then $E_d(p) = I(p,\hat{p})$, where $I(p,\hat{p})$ is Kullback's relative entropy [7] or directed divergence between the two distribution p and $\hat{p}$.

So, it will be natural if we call our function $E_d(p)$ as ***weighted relative entropy***. Note, that meaning of the coefficients d_i depend from fitness matrix $G = (g_{ij})$, and represents in an integrated form the value of different species groups.

It is obvious also that if the matrix $G = G^T$ then $d_i = 1, \quad i = 1,...,n,$ and for function (2) we obtain: $E(p) = p^T G p$, i.e. $E(p)$ is well-known Fisher's fitness, first introduced in population genetics by Fisher [4].

2. LYAPUNOV FUNCTION FOR GENERALIZED REPLICATOR SYSTEMS

Generalized replicator equations determine the evolution of probability distributions $p(t) = (p_1(t),...,p_n(t)) \in \sigma$ and have the next form [13]:

$$\dot{p}_i = h(p) f_i(p_i) \left(\sum_{j=1}^{n} w_{ij} f_j(p_j) - \theta^{-1}(p) \sum_{j,k=1}^{n} w_{jk} f_j(p_j) f_k(p_k) \right) i = 1,...,n \tag{4}$$

Here, f_i are nonlinear response functions satisfying the conditions $f_i(0) = 0$, $\partial f_i / \partial p_i > 0 \;\; p_i > 0$, and $\partial f_i / \partial p_i \geqslant 0$ for $p_i = 0$; $W = (w_{ij})$ is the matrix of interactions; the function $h:\sigma \to (0,\infty]$ is determined by the particular problem under consideration; $\theta(p) = \langle \mathbf{e}, f(p)\rangle$, where $\langle\cdot,\cdot\rangle$ is the inner product; and $f(p) = (f_1(p_1),..., f_n(p_n))$. Obviously, since $\langle \dot{p}(t), \mathbf{e}\rangle \equiv 0$ and $f_i(0) = 0$, the simplex σ and each of its faces are invariant sets of system (4).

System (4) has a very wide range of applications, from mathematical genetics to neural networks [2, 6]. Recently, it was shown [5] that system (4) can be obtained from Boltzmann-like equations. Thus, there are grounds for believing that system (4) determines the evolution of probability distributions for a fairly wide variety of processes.

To state the main theorem, we need some preliminary results. First, it is convenient to pass to the matrix form of representation. In this form, system (4) becomes

$$\dot{p} = h(p)\mathcal{D}(f)\left(Wf - \mathbf{e}\theta^{-1}(p)\langle f, Wf\rangle\right) \tag{5}$$

where $\mathcal{D}(f) = diag(f_1,..., f_n)$.

If the matrix W is nondegenerate, then system (5) has at most one isolated equilibrium point in $\mathrm{Int}\sigma$, which we call nontrivial.

Statement [13]. *System (5) has a unique nontrivial equilibrium point* $\hat{p} \in \mathrm{Int}\sigma$ *if and only if the vector* $W^{-1}\mathbf{e}$ *is either strictly positive or strictly negative.* □

The coordinates of this equilibrium point are determined from the system

$$\hat{f} \,/\, \langle \hat{f}, \mathbf{e}\rangle = W^{-1}\mathbf{e} \,/\, \langle \mathbf{e}, W^{-1}\mathbf{e}\rangle \tag{6}$$

where $\hat{f} = (f_1(\hat{p}_1),..., f_n(\hat{p}_n))$. Let us introduce the notation $W^{-1}\mathbf{e} = b$; then, for $\hat{p}_i$, we obtain the following system of $(n+1)$ equations:

$\hat{p}_i = f_i^{-1}(\lambda b_i) \qquad i = 1,...,n, \qquad \sum_{i=1}^{n} \hat{f}_i^{-1}(\lambda b_i) = 1$, where λ - is an auxiliary variable similar to the Lagrange multiplier. Obviously, this system has a unique solution if all the b_i are of the same sign.

Theorem 4 [13]. *If the matrix W is symmetric, then the function*

$$E(p) = \langle f(p), Wf(p)\rangle \theta^{-2}(p) \tag{7}$$

is a Lyapunov energy function for system (2). □

Corollary [13]. *If system (5) has a nontrivial equilibrium point* $\hat{p} \in \mathrm{Int}\sigma$ *, then it is totally stable in* $\mathrm{Int}\sigma$ *if and only if the matrix W has* $(n-1)$ *negative characteristic numbers.* □

Theorem 5.[13] *If* $W = W^T$ *and system (5) has a nontrivial equilibrium point* $\hat{p} \in \mathrm{Int}\sigma$ *which is totally stable in* $\mathrm{Int}\sigma$ *, then the entropy-like function:*

$$H(p) = \sum_{i=1}^{n} \int_{\hat{p}_i}^{p_i} \frac{\hat{f}_i dx}{f_i(x)} \qquad (8)$$

is a Lyapunov energy function for system (2), and

$$\dot{H} = h(p)\hat{\theta}\theta\left(E(\hat{p}) - E(p)\right) \geq 0. \quad \square \qquad (9)$$

This inequality is obvious, because the function E increases and attains its maximum at the point $\hat{p}$, which proves the theorem.

Now, we can state the main result without restriction $W = W^T$:

Theorem 6. Pykh [15] *If system (5) has a nontrivial equilibrium point* $\hat{p} \in Int\sigma$ *and the matrix* $(W^T + W)$ *has* $(n-1)$ *negative characteristic value, then the function*

$$H(p) = \sum_{i=1}^{n} \int_{\hat{p}_i}^{p_i} \frac{\hat{f}_i dx}{f_i(x)}$$

is a Lyapunov energy function for sytem (5). □

Based on this theorem we can recive a set of response function for existing entropy measures and constract new entropy and distance measures for any response functions. Short summary of this approach listed below in tables 1 and 2.

TABLE 1.
Different entropy measures

Response function	Entropy	Name
Logariphmic $f_i(p_i) = (1 - \ln p_i)^{-1}$	$H(p) = \sum_{i=1}^{n} p_i \ln p_i$	Bolzmann entropy
Power-law $f_i(p_i) = p_i^{1-q}$; $q \neq 1$	$H(p) = \frac{\left(\sum p_i^q - 1\right)}{1-q}$	Tsallis entropy
Logistic $f_i(p_i) = \frac{1}{b + ce^{-\alpha p_i}}$ $b > 0,\ c > 0,\ \alpha > 0$	$H(p) = \sum_{i=1}^{n} \ln\left(1 - e^{-\alpha p_i}\right)$	Logistic entropy (new)

TABLE 2.
Different distance measure

Response function	«Distanse »	Name
linear $f_i(p_i) = p_i$	$H = \sum \hat{p}_i \ln \frac{p_i}{\hat{p}_i}$	Relative entropy
linear $f_i(p_i) = p_i$	$H = \sum_{i=1}^{n} d_i \hat{p}_i \ln \frac{p_i}{\hat{p}_i} - \langle d, \hat{p} \rangle \ln \langle d, p \rangle$	Weighted relative entropy (Pykh [11,12])
logistic $f_i(p_i) = \frac{1}{b + ce^{-\alpha p_i}}$ $b > 0, c > 0,$ $\alpha > 0$	$H = \sum_{i=1}^{n} \hat{f}_i \ln\left(\frac{1 - e^{-\alpha p_i}}{1 - e^{-\alpha \hat{p}_i}}\right)$	Weighted logistic entropy (new)

3. OTHERS THERMODYNAMICS CHARACTERISTICS

We have received expression (2) for replicator's systems energy, expression (8) for systems entropy and expression (9) for entropy production. On the analogy of termodynamics laws, we can receive expression for systems temperature. Indeed if we redraft (9) as follows:

$$\frac{dH}{dE} = h(p)\hat{\theta}\theta(p),$$

then according to Clausius definition the systems temperature T is equal:

$$T = \left(h(\hat{p})\hat{\theta}\theta(p)\right)^{-1}$$

Note that in this case the temperature depends from systems steady-state. Now let us consider the exergy of the system.

Exergy is a measurement of how far a certain system deviates from a state of equilibrium with its environment. Exergy for a system in an environment usually is written as:

$$Ex = T\left(\hat{H} - H\right)$$

So we have a lot of different expression for exergy dependance from entropy i.e. from response function. If we put:

$$f_i(p_i) = \frac{1}{1 - \ln \frac{p_i}{\alpha_i}}$$

where vector $\alpha = (\alpha_1, ..., \alpha_n) \in \sigma$ and interaction matrix is stochastic i.e. $W\mathbf{e} = \mathbf{e}$, then $\alpha_i = \hat{p}_i$. In this case we receive the next expression for exergy:

$$Ex = h(p)\hat{\theta}\theta(p)\left(\sum_{i=1}^{n} p_i \ln \frac{p_i}{\hat{p}_i} + (p_i - \hat{p}_i)\right)$$

It is easy to see that this expression almost coincide with formula proposed by Mejer and Jorgensen in 1979. Note, that in like manner we can receive all termodynamic potentials such as Helmholtz or Gibbs free energy, which is also Lyapunov, functions.

4. CONCLUSION

It is seen from the examples given above that many (and practically all) known entropy and distance measures my be obtain from entropy-like Lyapunov function. We also emphasize that there exists a relation between the derivative of the function $H(p)$, which can be interpreted as a generalized entropy, and the function $E(p)$, which is often (in particular, in works on the theory of neural networks) considered as an analog of the energy or fitness. This relationship for entropy production was establihed by Pykh [14,15] for different interactions matrix and has the next form:

$$\dot{H}_{\varepsilon} = h(p)\hat{\theta}\theta\left(E(\hat{p}) - E(p)\right) \geqslant 0$$

In conclusion, we mention that all results stated above were obtained by formally analyzing systems of generalized replicator equations, which arise in very diverse fields of natural sciences and, therefore, can serve as a basis for finding analogies between these domains of natural sciences.

Also note that it was Ilya Prigogine who the first pointed out [9] the imoprtance of the relationship between Lyapunov functions and entropy.

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